Supplementary note: A Density Evolution Framework for Recovery of Covariance and Causal Graphs from Compressed Measurements

APPENDIX A
DERIVATION OF DE UPDATE EQUATIONS

As described in section 3, in order to analyze the convergence of the message-passing algorithm, the two quantities given by equations (8) and (9) are tracked over the course of the algorithm, re-written here for convenience.

$$E^{(t)} = \frac{1}{d^2p^2}\sum_{a=1}^{d^2}\sum_{i=1}^{p^2}\left(\mu_{i\to a}^{(t)} - \chi_i^*\right)^2;$$

$$V^{(t)} = \frac{1}{d^2p^2}\sum_{a=1}^{d^2}\sum_{i=1}^{p^2} v_{i\to a}^{(t)}.$$

To simplify these two quantities, we need to simplify the messages flowing through the factor graph. To that end, we start with the messages sent from the check nodes to the variable nodes, $\hat{m}_{a\to i}^{(t)} \sim \mathcal{N}\left(\hat{\mu}_{a\to i}^{(t)}, \hat{v}_{a\to i}^{(t)}\right)$. [1] derived a simplified update for the $\hat{\mu}_{a\to i}^{(t)}$ and $\hat{v}_{a\to i}^{(t)}$ in Lemma 6. Here we list the lemma and modify it our purpose to account for the Kronecker product sensing matrix.

Lemma 1: Consider the message flowing from check node a to variable node i, $\hat{m}_{a\to i}^{(t)} \sim \mathcal{N}\left(\hat{\mu}_{a\to i}^{(t)}, \hat{v}_{a\to i}^{(t)}\right)$. Then the following update can be obtained at the $(t+1)$-th iteration.

$$\hat{\mu}_{a\to i}^{(t+1)} = \chi_i + A\sum_{j\in\partial a\setminus i} A_{ai}^{\otimes}A_{aj}^{\otimes}\left(\chi_j - \mu_{j\to a}^{(t)}\right) + AA_{ai}^{\otimes}n_a; \tag{1}$$

$$\hat{v}_{a\to i}^{(t+1)} = A\sigma^2 + |\partial a|V^{(t)}. \tag{2}$$

Where χ_i is the i-th variable node and $|\partial a|$ is the degree of the check node a.

Now consider the message going from variable nodes to check nodes, $m_{i\to a}^{(t)} \sim \mathcal{N}\left(\mu_{i\to a}^{(t)}, v_{i\to a}^{(t)}\right)$. Using the previous lemma and exploiting some properties of Gaussian distribution with some approximations along the way, $\mu_{i\to a}^{(t)}$ and $v_{i\to a}^{(t)}$ can be updated as follows, here we also make use of the characterization of degrees of check nodes and the variable nodes from the section A. The readers are referred to [1] for more details.

$$\mu_{i\to a}^{(t+1)} \approx h_{\text{mean}}\left(\chi_i + z\sum_{i,i',j,j'} f^r_{i,i',j,j'}\sqrt{\mathcal{F}^r_{i,i',j,j'}};\ \sum_{i,i',j,j'} f^r_{i,i',j,j'}\mathcal{F}^r_{i,i',j,j'}\right); \tag{3}$$

$$v_{i\to a}^{(t+1)} \approx h_{\text{var}}\left(\chi_i + z\sum_{i,i',j,j'} f^r_{i,i',j,j'}\sqrt{\mathcal{F}^r_{i,i',j,j'}};\ \sum_{i,i',j,j'} f^r_{i,i',j,j'}\mathcal{F}^r_{i,i',j,j'}\right). \tag{4}$$

Where h_{mean} and h_{var} are given by,

$$h_{\text{mean}}(\mu; v) = \lim_{\beta\to\infty}\frac{\int x_i e^{-\beta f(x_i)}e^{-\frac{\beta(x_i-\mu)^2}{2v}}dx_i}{\int e^{-\beta f(x_i)}e^{-\frac{\beta(x_i-\mu)^2}{2v}}dx_i};$$

$$h_{\text{var}}(\mu; v) = \lim_{\beta\to\infty}\frac{\int x_i^2 e^{-\beta f(x_i)}e^{-\frac{\beta(x_i-\mu)^2}{2v}}dx_i}{\int e^{-\beta f(x_i)}e^{-\frac{\beta(x_i-\mu)^2}{2v}}dx_i} - h_{\text{mean}}(\mu; v)$$

By plugging equations (21) and (22) in (8) and (9) yields the following,

$$E^{(t+1)} = \mathbf{E}_{\text{prior}(s)}\mathbf{E}_z\left[h_{\text{mean}}\left(s + z\sum_{i,i',j,j'} f^r_{i,i',j,j'}\sqrt{\mathcal{F}^r_{i,i',j,j'}};\ \sum_{i,i',j,j'} f^r_{i,i',j,j'}\sqrt{\mathcal{F}^r_{i,i',j,j'}}\right) - s\right]^2; \tag{5}$$

$$V^{(t+1)} = \mathbf{E}_{\text{prior}(s)}\mathbf{E}_z h_{\text{var}}\left(s + z\sum_{i,i',j,j'} f^r_{i,i',j,j'}\sqrt{\mathcal{F}^r_{i,i',j,j'}};\ \sum_{i,i',j,j'} f^r_{i,i',j,j'}\sqrt{\mathcal{F}^r_{i,i',j,j'}}\right). \tag{6}$$

By setting $f(\boldsymbol{\chi}) = \beta\|\boldsymbol{\chi}\|_1$, we enforce the returned solutions to be sparse. This is equivalent to choosing Laplacian prior for $\boldsymbol{\chi}$. Following [2] in the noiseless case, equations (23) and (24) reduce to equations (10) and (11).

APPENDIX B
RELAXATION OF MESSAGE-PASSING CONVERGENCE CONSTRAINT

In this section we sketch the proof of Theorem 3.1, refer to [1] for more details of the proof. The derivation of necessary conditions for $\lim_{t\to\infty}(E^{(t)}, V^{(t)}) = (0, 0)$ can be split into two parts:

- **Part 1**. Showing that $(0, 0)$ is a fixed point of the DE update equation.
- **Part 2**. Necessary conditions for DE update equations to converge in the neighborhood of $(0, 0)$.

By substituting $(E^{(t)}, V^{(t)}) = (0, 0)$ we can see that it is indeed a fixed point. We begin part 2 by analyzing the functions $\delta_E^{(t)} = E^{(t+1)} - E^{(t)}$ and $\delta_V^{(t)} = V^{(t+1)} - V^{(t)}$. Let us define the functions Ψ_E and Ψ_V as follows,

$$\Psi_E(E^{(t)}; V^{(t)}) = \mathbf{E}_{\text{prior}(s)}\mathbf{E}_{z\sim\mathcal{N}(0,1)}\left[\text{prox}\Big(s + a_1 z\sqrt{E^{(t)}}; \beta a_2 V^{(t)}\Big) - s\right]^2;$$

$$\Psi_V(E^{(t)}; V^{(t)}) = \mathbf{E}_{\text{prior}(s)}\mathbf{E}_{z\sim\mathcal{N}(0,1)}\left[\beta a_2 V^{(t)} \text{prox}'\Big(s + a_1 z\sqrt{E^{(t)}}; \beta a_2 V^{(t)}\Big)\right]^2.$$

Taking the Taylor expansion of $\delta_E^{(t+1)}$ and $\delta_V^{(t+1)}$ and dropping the higher order terms we obtain,

$$\begin{bmatrix}\delta_E^{(t+1)}\\ \delta_V^{(t+1)}\end{bmatrix} = \underbrace{\begin{bmatrix}\left(\frac{\partial\Psi_E(E,V)}{\partial E}\right)^{(t)} & \left(\frac{\partial\Psi_E(E,V)}{\partial V}\right)^{(t)}\\ \left(\frac{\partial\Psi_V(E,V)}{\partial E}\right)^{(t)} & \left(\frac{\partial\Psi_V(E,V)}{\partial V}\right)^{(t)}\end{bmatrix}}_{=:\boldsymbol{L}^{(t)}}\begin{bmatrix}\delta_E^{(t)}\\ \delta_V^{(t)}\end{bmatrix}$$

For Ψ_E and Ψ_V to converge to 0, we would want the operator norm of $\boldsymbol{L}^{(t)}$ to be less than 1, i.e., $\inf_t \left\|\boldsymbol{L}^{(t)}\right\| \leq 1$. Since

$$\left\|\boldsymbol{L}^{(t)}\right\| = \max\left[\Big(\frac{\partial\Psi_E(E,V)}{\partial E}\Big)^{(t)}, \Big(\frac{\partial\Psi_V(E,V)}{\partial V}\Big)^{(t)}\right].$$

We can restrict the lower bounds of the individual terms to be less than 1. This would result in

$$a_1^2 \leq \frac{p^2}{k^2}, \quad a_2 \leq \frac{p^2}{k^2\beta}.$$

Appendix C
Relaxation of Constraints for Preferential Sensing

In this section, we provide details for the relaxation of requirements (2) and (3) for preferential sensing. In this regime, we separately track the average error and the variance of the HH, HL (LH), and LL parts of the covariance matrix separately. The quantities $E_{HH}^{(t)}, V_{HH}^{(t)}, E_{HL}^{(t)}, V_{HL}^{(t)}$, and $E_{LL}^{(t)}, V_{LL}^{(t)}$ are defined as described in section IV-A and following the procedure described in appendix A yields equations (19) and (20) from the main text. Let us now define the following quantities

$$\begin{aligned}
E_{HH}^{(t+1)} &= \mathbf{E}_{\text{prior}(s)}\mathbf{E}_{z\sim\mathcal{N}(0,1)}\left[\text{prox}\Big(s + zb_{HH,1}^{(t)}; b_{HH,2}^{(t)}\Big) - s\right]^2 \\
&\quad \Psi_{E,HH}\Big(E_{HH}^{(t)}, V_{HH}^{(t)}, E_{HL}^{(t)}, V_{HL}^{(t)}, E_{LL}^{(t)}, V_{LL}^{(t)}\Big);\\
E_{HL}^{(t+1)} &= \mathbf{E}_{\text{prior}(s)}\mathbf{E}_{z\sim\mathcal{N}(0,1)}\left[\text{prox}\Big(s + zb_{HL,1}^{(t)}; b_{HL,2}^{(t)}\Big) - s\right]^2 \\
&\quad \Psi_{E,HL}\Big(E_{HH}^{(t)}, V_{HH}^{(t)}, E_{HL}^{(t)}, V_{HL}^{(t)}, E_{LL}^{(t)}, V_{LL}^{(t)}\Big);\\
E_{LL}^{(t+1)} &= \mathbf{E}_{\text{prior}(s)}\mathbf{E}_{z\sim\mathcal{N}(0,1)}\left[\text{prox}\Big(s + zb_{LL,1}^{(t)}; b_{LL,2}^{(t)}\Big) - s\right]^2 \\
&\quad \Psi_{E,LL}\Big(E_{HH}^{(t)}, V_{HH}^{(t)}, E_{HL}^{(t)}, V_{HL}^{(t)}, E_{LL}^{(t)}, V_{LL}^{(t)}\Big);
\end{aligned}$$

Similarly,

$$\begin{aligned}
V_{HH}^{(t+1)} &= \mathbf{E}_{\text{prior}(s)}\mathbf{E}_{z\sim\mathcal{N}(0,1)}\left[b_{HH,2}^{(t)}\text{prox}'\Big(s + zb_{HH,1}^{(t)}; b_{HH,2}^{(t)}\Big)\right] \\
&\quad \Psi_{V,HH}\Big(E_{HH}^{(t)}, V_{HH}^{(t)}, E_{HL}^{(t)}, V_{HL}^{(t)}, E_{LL}^{(t)}, V_{LL}^{(t)}\Big);\\
V_{HL}^{(t+1)} &= \mathbf{E}_{\text{prior}(s)}\mathbf{E}_{z\sim\mathcal{N}(0,1)}\left[b_{HH,2}^{(t)}\text{prox}'\Big(s + zb_{HL,1}^{(t)}; b_{HL,2}^{(t)}\Big)\right] \\
&\quad \Psi_{V,HL}\Big(E_{HH}^{(t)}, V_{HH}^{(t)}, E_{HL}^{(t)}, V_{HL}^{(t)}, E_{LL}^{(t)}, V_{LL}^{(t)}\Big);\\
V_{LL}^{(t+1)} &= \mathbf{E}_{\text{prior}(s)}\mathbf{E}_{z\sim\mathcal{N}(0,1)}\left[b_{HH,2}^{(t)}\text{prox}'\Big(s + zb_{LL,1}^{(t)}; b_{LL,2}^{(t)}\Big)\right] \\
&\quad \Psi_{V,HL}\Big(E_{HH}^{(t)}, V_{HH}^{(t)}, E_{HL}^{(t)}, V_{HL}^{(t)}, E_{LL}^{(t)}, V_{LL}^{(t)}\Big);
\end{aligned}$$

We now define $\delta_{E,HH}^{(t)}, \delta_{E,HL}^{(t)}, \delta_{E,LL}^{(t)}$, and $\delta_{V,HH}^{(t)}, \delta_{V,HL}^{(t)}, \delta_{V,LL}^{(t)}$ in a similar manner to that in appendix B.

A. Relaxation of Requirement 2

We use the shorthand, $\Psi_{V,HH}^{(t)} = \Psi_{V,HL}\Big(E_{HH}^{(t)}, V_{HH}^{(t)}, E_{HL}^{(t)}, V_{HL}^{(t)}, E_{LL}^{(t)}, V_{LL}^{(t)}\Big)$ for ease of notation. Approximate $\delta_{V,HH}^{(t)}$ using its First-order Taylor series expansion, we get

$$\begin{aligned}
\delta_{V,HH}^{(t+1)} &= \Psi_{V,HH}^{(t+1)} - \Psi_{V,HH}^{(t)}\\
&= \left(\frac{\partial\Psi_{V,HH}(\cdot)}{\partial E_{HH}}\right)^{(t)}\delta_{E,HH}^{(t)} + \left(\frac{\partial\Psi_{V,HH}(\cdot)}{\partial E_{HL}}\right)^{(t)}\delta_{E,HL}^{(t)}\\
&\quad + \left(\frac{\partial\Psi_{V,HH}(\cdot)}{\partial E_{LL}}\right)^{(t)}\delta_{E,LL}^{(t)} + \left(\frac{\partial\Psi_{V,HH}(\cdot)}{\partial V_{HH}}\right)^{(t)}\delta_{V,HH}^{(t)}\\
&\quad + \left(\frac{\partial\Psi_{V,HH}(\cdot)}{\partial V_{HL}}\right)^{(t)}\delta_{V,HL}^{(t)} + \left(\frac{\partial\Psi_{V,HH}(\cdot)}{\partial V_{LL}}\right)^{(t)}\delta_{V,LL}^{(t)}\\
&\quad + O\Big(\big(\delta_{V,HH}^{(t)}\big)^2\Big) + O\Big(\big(\delta_{V,HL}^{(t)}\big)^2\Big) + O\Big(\big(\delta_{V,LL}^{(t)}\big)^2\Big)
\end{aligned}$$

Following the same template as appendix B, the derivation consists of two parts:

I. Verify that $(0, 0, 0)$ is a fixed point. Which is a trivial task.
II. Show that the DE equations w.r.t to $V_{HH}^{(t)}, V_{HL}^{(t)}, V_{LL}^{(t)}$ converges within a proximity of the origin.

It can be trivially checked that part I is true. We now focus our attention to part II. Consider the region where $V_{HH}^{(t)}, V_{HL}^{(t)}, V_{LL}^{(t)}$, in this case, we can ignore the quadratic terms in the above equation. By exploiting the fact that $\partial\Psi_{V,HH}/\partial E_{HH} = \partial\Psi_{V,HH}/\partial E_{HL} = \partial\Psi_{V,HH}/\partial E_{LL} = 0$, we obtain the following.

$$\begin{bmatrix}\delta_{V,HH}^{(t+1)}\\ \delta_{V,HL}^{(t+1)}\\ \delta_{V,LL}^{(t+1)}\end{bmatrix} = \underbrace{\begin{bmatrix}\left(\frac{\partial\Psi_{V,HH}}{\partial V_{HH}}\right)^{(t)} & \left(\frac{\partial\Psi_{V,HH}}{\partial V_{HL}}\right)^{(t)} & \left(\frac{\partial\Psi_{V,HH}}{\partial V_{LL}}\right)^{(t)}\\ \left(\frac{\partial\Psi_{V,HL}}{\partial V_{HH}}\right)^{(t)} & \left(\frac{\partial\Psi_{V,HL}}{\partial V_{HL}}\right)^{(t)} & \left(\frac{\partial\Psi_{V,HL}}{\partial V_{LL}}\right)^{(t)}\\ \left(\frac{\partial\Psi_{V,LL}}{\partial V_{HH}}\right)^{(t)} & \left(\frac{\partial\Psi_{V,LL}}{\partial V_{HL}}\right)^{(t)} & \left(\frac{\partial\Psi_{V,LL}}{\partial V_{LL}}\right)^{(t)}\end{bmatrix}}_{\boldsymbol{L}_V^{(t)}} \times \begin{bmatrix}\delta_{V,HH}^{(t)}\\ \delta_{V,HL}^{(t)}\\ \delta_{V,LL}^{(t)}\end{bmatrix}$$

To make the LHS convergent we require $\inf_t \left\|\boldsymbol{L}_V^{(t)}\right\|_{OP} \leq 1$. We now lower each term in the first row of $\boldsymbol{L}_V^{(t)}$ similar to what was done in appendix B, hence we omit the details. We then obtain,

$$\left(\frac{\partial\Psi_{V,HH}}{\partial V_{HH}}\right)^{(t)} \geq \frac{k_{HH}\beta_{HH}}{n_{HH}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\ell}\right)^2\left(\sum_i i\rho_{H,i}\right)^2$$
$$\left(\frac{\partial\Psi_{V,HH}}{\partial V_{HL}}\right)^{(t)} \geq \frac{k_{HH}\beta_{HH}}{n_{HH}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\ell}\right)^2\left(\sum_i i\rho_{H,i}\right)\left(\sum_j j\rho_{L,j}\right)$$
$$\left(\frac{\partial\Psi_{V,HH}}{\partial V_{LL}}\right)^{(t)} \geq \frac{k_{HH}\beta_{HH}}{n_{HH}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\ell}\right)^2\left(\sum_i i\rho_{L,i}\right)^2$$

Following the same procedure for the second row, we get

$$\left(\frac{\partial\Psi_{V,HL}}{\partial V_{HH}}\right)^{(t)} \geq \frac{k_{HL}\beta_{HL}}{n_{HL}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\ell}\right)\left(\sum_k \frac{\lambda_{L,k}}{k}\right)\left(\sum_i i\rho_{H,i}\right)^2$$
$$\left(\frac{\partial\Psi_{V,HL}}{\partial V_{HL}}\right)^{(t)} \geq \frac{k_{HL}\beta_{HL}}{n_{HL}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\ell}\right)\left(\sum_k \frac{\lambda_{L,k}}{k}\right)\left(\sum_i i\rho_{H,i}\right)\left(\sum_j j\rho_{L,j}\right)$$
$$\left(\frac{\partial\Psi_{V,HL}}{\partial V_{LL}}\right)^{(t)} \geq \frac{k_{HL}\beta_{HL}}{n_{HL}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\ell}\right)\left(\sum_k \frac{\lambda_{L,k}}{k}\right)\left(\sum_i i\rho_{L,i}\right)^2$$

And finally for row 3 we get,

$$\left(\frac{\partial\Psi_{V,LL}}{\partial V_{HH}}\right)^{(t)} \geq \frac{k_{LL}\beta_{LL}}{n_{LL}}\left(\sum_\ell \frac{\lambda_{L,\ell}}{\ell}\right)^2\left(\sum_i i\rho_{H,i}\right)^2$$
$$\left(\frac{\partial\Psi_{V,LL}}{\partial V_{HL}}\right)^{(t)} \geq \frac{k_{LL}\beta_{LL}}{n_{LL}}\left(\sum_\ell \frac{\lambda_{L,\ell}}{\ell}\right)^2\left(\sum_i i\rho_{H,i}\right)\left(\sum_j j\rho_{L,j}\right)$$
$$\left(\frac{\partial\Psi_{V,LL}}{\partial V_{LL}}\right)^{(t)} \geq \frac{k_{LL}\beta_{LL}}{n_{LL}}\left(\sum_\ell \frac{\lambda_{L,\ell}}{\ell}\right)^2\left(\sum_i i\rho_{L,i}\right)^2$$

Equation (26) from the main text is then obtained by enforcing the condition on the operator norm on the above inequalities.

B. Relaxation of Requirement 3

The basic idea remains the same as in the previous subsection. We linearize the DE update equation with Taylor expansion and enforce the difference $\delta_{E,HH}^{(t)}$ to decrease faster than $\delta_{E,HL}^{(t)}$ and $\delta_{E,LL}^{(t)}$. That is,

$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{HH}}\right)^{(t)} \leq \left(\frac{\partial\Psi_{E,HL}}{\partial E_{HH}}\right)^{(t)}; \quad (7)$$
$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{HL}}\right)^{(t)} \leq \left(\frac{\partial\Psi_{E,HL}}{\partial E_{HL}}\right)^{(t)}; \quad (8)$$
$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{LL}}\right)^{(t)} \leq \left(\frac{\partial\Psi_{E,HL}}{\partial E_{LL}}\right)^{(t)}. \quad (9)$$

And,

$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{HH}}\right)^{(t)} \leq \left(\frac{\partial\Psi_{E,LL}}{\partial E_{HH}}\right)^{(t)}; \quad (10)$$
$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{HL}}\right)^{(t)} \leq \left(\frac{\partial\Psi_{E,LL}}{\partial E_{HL}}\right)^{(t)}; \quad (11)$$
$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{LL}}\right)^{(t)} \leq \left(\frac{\partial\Psi_{E,LL}}{\partial E_{LL}}\right)^{(t)}. \quad (12)$$

Following the same logic as the previous subsection, we can lower-bound each of the gradients in the above inequalities. We then obtain,

$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{HH}}\right)^{(t)} \geq \frac{k_{HH}}{n_{HH}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\sqrt{\ell}}\right)^4\left(\sum_i \sqrt{i}\rho_{H,i}\right)^4$$
$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{HL}}\right)^{(t)} \geq \frac{k_{HH}}{n_{HH}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\sqrt{\ell}}\right)^4\left(\sum_i \sqrt{i}\rho_{H,i}\right)^2\left(\sum_j \sqrt{j}\rho_{L,j}\right)^2$$
$$\left(\frac{\partial\Psi_{E,HH}}{\partial E_{LL}}\right)^{(t)} \geq \frac{k_{HH}}{n_{HH}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\sqrt{\ell}}\right)^4\left(\sum_i \sqrt{i}\rho_{L,i}\right)^4$$

And,

$$\left(\frac{\partial\Psi_{E,HL}}{\partial E_{HH}}\right)^{(t)} \geq \frac{k_{HL}}{n_{HL}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\sqrt{\ell}}\right)^2\left(\sum_\ell \frac{\lambda_{L,\ell}}{\sqrt{\ell}}\right)^2\left(\sum_i \sqrt{i}\rho_{H,i}\right)^4$$
$$\left(\frac{\partial\Psi_{E,HL}}{\partial E_{HL}}\right)^{(t)} \geq \frac{k_{HL}}{n_{HL}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\sqrt{\ell}}\right)^2\left(\sum_\ell \frac{\lambda_{L,\ell}}{\sqrt{\ell}}\right)^2\left(\sum_i \sqrt{i}\rho_{H,i}\right)^2\left(\sum_j \sqrt{j}\rho_{L,j}\right)^2$$
$$\left(\frac{\partial\Psi_{E,HL}}{\partial E_{LL}}\right)^{(t)} \geq \frac{k_{HL}}{n_{HL}}\left(\sum_\ell \frac{\lambda_{H,\ell}}{\sqrt{\ell}}\right)^2\left(\sum_\ell \frac{\lambda_{L,\ell}}{\sqrt{\ell}}\right)^2\left(\sum_i \sqrt{i}\rho_{L,i}\right)^4$$

Finally,

$$\left(\frac{\partial\Psi_{E,LL}}{\partial E_{HH}}\right)^{(t)} \geq \frac{k_{LL}}{n_{LL}}\left(\sum_\ell \frac{\lambda_{L,\ell}}{\sqrt{\ell}}\right)^4\left(\sum_i \sqrt{i}\rho_{H,i}\right)^4$$
$$\left(\frac{\partial\Psi_{E,LL}}{\partial E_{HL}}\right)^{(t)} \geq \frac{k_{LL}}{n_{LL}}\left(\sum_\ell \frac{\lambda_{L,\ell}}{\sqrt{\ell}}\right)^4\left(\sum_i \sqrt{i}\rho_{H,i}\right)^2\left(\sum_j \sqrt{j}\rho_{L,j}\right)^2$$
$$\left(\frac{\partial\Psi_{E,LL}}{\partial E_{LL}}\right)^{(t)} \geq \frac{k_{LL}}{n_{LL}}\left(\sum_\ell \frac{\lambda_{L,\ell}}{\sqrt{\ell}}\right)^4\left(\sum_i \sqrt{i}\rho_{L,i}\right)^4$$

Combining this with inequalities (38)-(43) yields the inequalities (27) and (28) in the main text.

REFERENCES

[1] H. Zhang, A. Abdi, and F. Fekri, "a general compressive sensing construct using density evolution," *IEEE Transactions on Signal Processing*, pp. 1–16, 2022.

[2] D. L. Donoho, A. Maleki, and A. Montanari, "Message-passing algorithms for compressed sensing," *Proceedings of the National Academy of Sciences*, vol. 106, no. 45, pp. 18 914–18 919, 2009.